Erotetic Search Scenarios as families of sequences and Erotetic Search Scenarios as trees: two different, yet equal accounts. Research Report

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Dorota Leszczyńska-Jasion
Department of Logic and Cognitive Science
Institute of Psychology, Adam Mickiewicz University

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Poznań, Poland
www.kognitywistyka.amu.edu.pl/intquestpro

Erotetic Search Scenarios as families of sequences and Erotetic Search Scenarios as trees: two different, yet equal accounts (Research Report)

Dorota Leszczyńska-Jasion

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Abstract

Our aim is to show that two different accounts of erotetic search scenarios — namely, e-scenarios defined as families of interconnected sequences and e-scenarios defined as labelled trees — may be considered as equal.

Keywords: erotetic search scenarios

1 Introduction

The idea of erotetic search scenarios has been first presented in [2] and [1]. The central concept — that of an erotetic search scenario (e-scenario, for short) — has been defined in terms of sequences there, namely, as a family of interconnected sequences (the so-called erotetic derivations). However, these objects may be also viewed as labelled trees (see [3] for the latest account).[1] The former concept occurs to be much more "handy" in proving things about scenarios, whereas the later seems to have greater explanatory value.

Below we will consider e-scenarios defined as labelled trees[2] and these defined as families of sequences and we will show that these objects may be considered as equivallent (or at least, representing certain idea in equally satisfying way), although they obviously are not isomorphic.

We will, however, recall the isomorphism idea at the start, since we do not want to consider too many labelled trees. Namely, if two labelled trees are indistinguishable with respect to their structure and labels, then we will assume that there is only one tree. (Thus we will neglect the nature of the nodes, so to say.)

[1] They may be also considered as trees without the labelling function.

[2] Labelled trees are assumed to be triples $< X, R, \ell >$, where R is a partial order on X, X has the smallest element with respect to R and each node has at most one immediate R-predecessor. ℓ is a labelling function, that is, a function assigning to each node from X a label.

2 Definitions

The following definitions come from [3]. (See also [2] and [1].)

DEFINITION 1 (***E-derivation***)**.** *A finite sequence* $\mathtt{s} = \mathtt{s}_1, \ldots, \mathtt{s}_n$ *of wffs is an erotetic derivation (e-derivation for short) of a direct answer A to question Q on the basis of a set of d-wffs X iff* $\mathtt{s}_1 = Q$, $\mathtt{s}_n = A$, *and the following conditions hold:*

(1) *for each question* $\mathtt{s}_k$ *of* $\mathtt{s}$ *such that* $k > 1$*:*

- **(a)** $\mathbf{d}\mathtt{s}_k \neq \mathbf{d}Q$,
- **(b)** $\mathtt{s}_k$ *is implied by a certain question* $\mathtt{s}_j$ *which precedes* $\mathtt{s}_k$ *in* $\mathtt{s}$ *on the basis of the empty set, or on the basis of a non-empty set of d-wffs such that each element of this set precedes* $\mathtt{s}_k$ *in* $\mathtt{s}$, *and*
- **(c)** $\mathtt{s}_{k+1}$ *is either a direct answer to* $\mathtt{s}_k$ *or a question;*

(2) *for each d-wff* $\mathtt{s}_i$ *of* $\mathtt{s}$*:*

- **(a)** $\mathtt{s}_i \in X$, *or*
- **(b)** $\mathtt{s}_i$ *is a direct answer to* $\mathtt{s}_{i-1}$, *where* $\mathtt{s}_{i-1} \neq Q$, *or*
- **(c)** $\mathtt{s}_i$ *is entailed by a certain non-empty set of d-wffs such that each element of this set precedes* $\mathtt{s}_i$ *in* $\mathtt{s}$*;*

Below, by an *e-t-scenario* we will mean an e-scenario defined as a tree (see Definition 3), and by an *e-s-scenario* we will mean an e-scenario defined as a family of sequences (Definition 2).

DEFINITION 2 (***E-s-scenario***)**.** *A finite family* Σ *of sequences of wffs is an erotetic search scenario (e-scenario for short) for a question Q relative to a set of d-wffs X iff each element of* Σ *is an e-derivation of a direct answer to Q on the basis of X and the following conditions hold:*

(1) $\mathbf{d}Q \cap X = \emptyset$*;*

(2) Σ *contains at least two elements;*

(3) *for each element* $\mathtt{s} = \mathtt{s}_1, \ldots, \mathtt{s}_n$ *of* Σ, *for each index k, where* $1 \leq k < n$*:*

- **(a)** *if* $\mathtt{s}_k$ *is a question and* $\mathtt{s}_{k+1}$ *is a direct answer to* $\mathtt{s}_k$, *then for each direct answer B to* $\mathtt{s}_k$*: the family* Σ *contains a certain e-derivation* $\mathtt{s}^* = \mathtt{s}^*_1, \mathtt{s}^*_2, \ldots, \mathtt{s}^*_m$ *such that* $\mathtt{s}_j = \mathtt{s}^*_j$ *for* $j = 1, \ldots, k$, *and* $\mathtt{s}^*_{k+1} = B$*;*
- **(b)** *if* $\mathtt{s}_k$ *is a d-wff, or* $\mathtt{s}_k$ *is a question and* $\mathtt{s}_{k+1}$ *is not a direct answer to* $\mathtt{s}_k$, *then for each e-derivation* $\mathtt{s}^* = \mathtt{s}^*_1, \mathtt{s}^*_2, \ldots, \mathtt{s}^*_m$ *in* Σ *such that* $\mathtt{s}_j = \mathtt{s}^*_j$ *for* $j = 1, \ldots, k$ *we have* $\mathtt{s}_{k+1} = \mathtt{s}^*_{k+1}$.

And here is the alternative account:

DEFINITION 3 (***E-t-scenario***)**.** *A finite labeled tree* Φ *is an erotetic search scenario for a question Q relative to a set of d-wffs X iff*

(1) *the nodes of* Φ *are labeled by questions and d-wffs; they are called e-nodes and d-nodes, respectively;*

(2) *Q labels the root of Φ;*

(3) *each leaf of Φ is labeled by a direct answer to Q;*

(4) $\boldsymbol{d}Q \cap X = \emptyset$;

(5) *for each d-node γ_δ of Φ: if A is the label of γ_δ, then*

- **(a)** $A \in X$, *or*
- **(b)** $A \in \boldsymbol{d}Q^*$, *where $Q^* \neq Q$ and Q^* labels the immediate predecessor of γ_δ;*
- **(c)** $\{B_1, ..., B_n\} \models A$, *where B_i $(1 \leq i \leq n)$ labels a d-node of Φ that precedes the d-node γ_δ in Φ;*

(6) *each d-node of Φ has at most one immediate successor;*

(7) *there exists at least one e-node of Φ which is different from the root;*

(8) *for each e-node γ_ε of Φ different from the root: if Q^* is the label of γ_ε, then $\boldsymbol{d}Q^* \neq \boldsymbol{d}Q$ and*

- **(a)** $\mathbf{Im}(Q^{**}, Q^*)$ *or* $\mathbf{Im}(Q^{**}, B_1, ..., B_n, Q^*)$, *where Q^{**} labels an e-node of Φ that precedes γ_ε in Φ and B_i $(1 \leq i \leq n)$ labels a d-node of Φ that precedes γ_ε in Φ, and*
- **(b)** *an immediate successor of γ_ε is either an e-node or is a d-node labeled by a direct answer to the question that labels γ_ε, moreover*
  - *if an immediate successor of γ_ε is an e-node, it is the only immediate successor of γ_ε,*
  - *if an immediate successor of γ_ε is not an e-node, then for each direct answer to the question that labels γ_ε there exists exactly one immediate successor of γ_ε labeled by the answer.*

3 Proofs

Below we use the concept of *level of a node* **n** *in tree* **t**, which is understood as follows:

- the root of **t** is of level 1
- if **n** is of level k $(k \geq 1)$, then the immediate succesor of **n** (if there is any) is of level $k + 1$

We consider chains[3] which have the smallest element and a labelling function ℓ. For simplicity, if a chain has the smallest element, then we will call it an *r-chain* (where 'r' is for 'root') and if it goes with its labelling function, then we will call it a *labelled r-chain.* Each branch of a labelled tree may be considered as a labelled r-chain (if we restrict the labelling function of the tree to its branch). Moreover, each labelled r-chain is a labelled tree (with no branching points).

[3]By a *chain* we mean a lineary ordered set.

Our aim is to map labelled r-chains into sequences in a bijective way. Up to now the idea is clear, since our chains (i) have "begginings" (i.e. their roots), as sequences do, (ii) are "organized" (by the notion of level of a node), just as sequences are, and (iii) have labels which are to be identified with the elements of sequences. Therefore we consider function ϕ which mapps labelled r-chains into sequences in the following way:

DEFINITION 4. *Let* $\mathbf{c}$ *stand for a labelled r-chain, and let* $\mathbf{n}$ *– node,* ℓ *– labelling function. If a node* $\mathbf{n}$ *of* $\mathbf{c}$ *is of level* k*, then* $\ell(\mathbf{n})$ *is the* k*-th element of sequence* $\phi(\mathbf{c})$*.*

Function ϕ is a bijection. Thus if $\mathbf{s}$ is a sequence of wffs, then $\phi^{-1}(\mathbf{s})$ is an r-chain labelled with wffs. We will use the following observations:

COROLLARY 1. *Suppose that node* $\mathbf{n}$ *occurs on branch* $\mathbf{b}$ *of tree* $\mathbf{t}$ *and suppose that node* $\mathbf{m}$ *precedes* $\mathbf{n}$ *in* $\mathbf{t}$*. Then* $\ell(\mathbf{m})$ *precedes* $\ell(\mathbf{n})$ *in sequence* $\phi(\mathbf{b})$*. Similarly, if* s_i *precedes* s_k *in sequence* $\mathbf{s}$*, then* s_i *and* s_k *label two nodes of* $\phi^{-1}(\mathbf{s})$ *such that the node labelled with* s_i *precedes the node labelled by* s_k*.*

Let us also note:

COROLLARY 2. *Every e-t-scenario has at least two distinct branches.*

Proof. By clause (7) of Definition 3 every e-t-scenario has at least one e-node different from the root. By the fact that scenarios have d-nodes as leaves and by clause (8b) of the same definition it follows that either the e-node mentioned above is answered (and thus is a branching point) or it is succeded by another e-node which is a branching point. □

On the other hand, one may show that (cf. [2, p.410]):

COROLLARY 3. *Every e-s-scenario has at least one query, which is a branching point of the e-scenario.*

COROLLARY 4. *If* $\mathbf{s}$ *is an element (an e-derivation) of an e-s-scenario and for some* k *(*$1 \leq k < n$*)* s_k *is a question and* s_{k+1} *is a direct answer to it, then* s_k *is not the root.*

Proof. By the definition of an e-derivation, the root may be followed by a direct answer to it only when the direct answer is an element of X. This, however, is forbidden by clause (1) of Definition 2 (of an e-s-scenario). □

Now we may show:

COROLLARY 5. *Suppose that* $\mathbf{t}$ *is an e-t-scenario for question* Q *relative to* X*, let* $\mathbf{b}$ *be a branch of* $\mathbf{t}$ *and let* A *stand for the label of the leaf of* $\mathbf{b}$*. Then* $\phi(\mathbf{b})$ *is an e-derivation of direct answer* A *to question* Q *on the basis of* X*.*

Proof. By clause (1) of Definition 3, $\phi(\mathbf{b})$ is a (finite) sequence of wffs. Let $\phi(\mathbf{b}) =< s_1, \ldots, s_n >$. By clauses (2) and (3) of definition 3, $s_1 = Q$, $A \in \mathbf{d}Q$ and $s_n = A$. Assume that s_k (where $k > 1$) is a question. Then:

- $\mathbf{d}s_k \neq \mathbf{d}Q$ (by clause (8) of Definition 3)

- s_k is implied by some question which precedes s_k in $\phi(\mathbf{b})$ on the basis of the empty set, or on the basis of a set of d-wffs such that each element of this set precedes s_k in $\phi(\mathbf{b})$ (by clause (8a) of Definition 3 and by Corollary 1)
- s_{k+1} is either a question or a direct answer to s_k (by clause (8b) of Definition 3)

Now assume that s_k (where $k \geq 1$) is a d-wff. Then:

- $s_k \in X$ (clause (5a), Definition 3) or
- s_k is a direct answer to s_{k-1}, where $s_{k-1} \neq Q$ (clause (5b) of Definition 3) or
- s_k is entailed by a certain non-empty set of d-wffs such that each element of this set precedes s_k in $\phi(\mathbf{b})$ (clause (5c) of Definition 3 and Corollary 1).

□

And finally:

THEOREM 1. *Assume that* $\mathbf{t}$ *is an e-t-scenario for question* Q *relative to* X *and let* $\mathcal{B}$ *be the set of all branches of* $\mathbf{t}$*. Then* $\phi(\mathcal{B})$ *is an e-s-scenario for question* Q *relative to* X*.*

Proof. Clearly, $\mathcal{B}$ is a finite set, thus by Corollary 5 $\phi(\mathcal{B})$ is a finite family of e-derivations of direct answers to Q on the basis of X. Moreover:

- $\mathbf{d}Q \cap X = \emptyset$, by clause (4) of Definition 3,
- by Corollary 2, $\phi(\mathcal{B})$ contains at least two elements.

Now consider an arbitrary element $\mathbf{s} =< s_1, \ldots, s_n >$ of $\phi(\mathcal{B})$ and an arbitrary index k ($1 \leq k < n$).

- If s_k is a question and s_{k+1} is a direct answer to s_k, then by Corollary 4, s_k is not the root, and thus by clause (8b) of Definition 3, the k-th element of $\phi^{-1}(\mathbf{s})$ is a branching point of the e-t-scenario which is immediately succeeded by each of the direct answers to s_k (and only by them). And this amounts to the fact that clause (3a) of the definition of e-s-scenario is fulfilled.
- If it is not the case that s_k is a question followed by the direct answer to it, then by clauses (8b) and (6) of Definition 3, the k-th element of $\phi^{-1}(\mathbf{s})$ cannot be a branching point of the e-t-scenario, and thus clause (3b) of the definition of e-s-scenario is fulfilled.

□

Moreover,

COROLLARY 6. *Let* $\mathbf{t}$ *be an e-t-scenario and let* $\mathcal{B}$ *be the set of all branches of* $\mathbf{t}$*. Then:*

- *A wff, A, labels a leaf of* $\mathbf{t}$ *iff for some e-derivation in $\phi(\mathcal{B})$, A is its last element.*

- *Node* $\mathbf{n}$ *is a branching point of* $\mathbf{t}$ *iff $\ell(\mathbf{n})$ is a branching point of $\phi(\mathcal{B})$.*[4]

Let us observe, that after "dividing" a tree into branches we may "sum up" the branches in order to obtain the tree again. Namely,

COROLLARY 7. *If $\mathbf{t} =< X, R, \ell >$ is a finite tree*[5]*, and $\mathcal{B} = \{b_1, \ldots, b_n\}$ is the set of its branches, then:*

- *each branch b_i may be represented as $b_i =< X_i, R_i, \ell_i >$, since b_i is a tree itself,*

- *we have $\bigcup_{i=1}^{n} X_i = X$, $\bigcup_{i=1}^{n} R_i = R$ and $\bigcup_{i=1}^{n} \ell_i = \ell$,*

  and therefore

- $< \bigcup_{i=1}^{n} X_i, \bigcup_{i=1}^{n} R_i, \bigcup_{i=1}^{n} \ell_i >= \mathbf{t}$.

Now let us consider the following situation: we have a set, $\mathcal{B}$, of r-chains which have a common label of the root, but the chains need not come from the same tree. Now we need some more general way of "sticking" the branches together to form a tree, and this is how we do it.

DEFINITION 5. *Let $\mathcal{C}$ be a finite, nonempty set of finite r-chains labelled with wffs.*

1. *If the root of each chain in $\mathcal{C}$ is labelled with the same wff, then by $\mathbf{r}(\mathcal{C})$ we mean the label, and we say that the set $\mathcal{C}$ has a root, or that $\mathbf{r}(\mathcal{C})$ is the root of $\mathcal{C}$. Otherwise we say that $\mathcal{C}$ has no root.*

2. *By $\mathcal{C}-r$ we mean a finite (possibly empty) set of finite r-chains labelled with formulas which is obtained from $\mathcal{C}$ by (i) removing each singleton chain (if there is any) and (ii) for each non-singleton chain $\mathbf{b}$, $\mathbf{b}$ is replaced with a chain $\mathbf{b}'$ obtained from $\mathbf{b}$ by removing its root (and by restricting the ordering relation and the labelling function to the remaining set).*

Now for each finite set $\mathcal{B}$ of finite r-chains which has a root we may define a tree in the following way:

DEFINITION 6. *Let $\mathcal{B}$ stand for a finite set of finite r-chains (labelled with wffs) which has a root. Then by $\mathcal{T}(\mathcal{B})$ we mean a tree defined as follows:*

- *The root of $\mathcal{T}(\mathcal{B})$ is the whole set $\mathcal{B}$ and $\mathbf{r}(\mathcal{B})$ is the label of the root.*

- *Suppose that the tree is constructed up to level k, that is, each of its branches has at most k nodes and at least one branch has exactly k nodes. Then for each branch which has exactly k nodes, if set $\mathcal{C}$ is the leave of the branch, then*

[4] For the notion of a branching point of an e-s-scenario see [2, p. 410].

[5] An analogous corollary may be formulated with respect to infinite trees.

1. *if $\mathcal{C} - r$ is empty, then $\mathcal{C}$ has no immediate successors (that is, $\mathcal{C}$ is a leaf of tree $\mathcal{T}(\mathcal{B})$;*
2. *if $\mathcal{C} - r$ has a root, then $\mathcal{C} - r$ is the immediate successor of $\mathcal{C}$ in tree $\mathcal{T}(\mathcal{B})$ and $\mathbf{r}(\mathcal{C} - r)$ is the label of $\mathcal{C} - r$;*
3. *if $\mathcal{C} - r$ is nonempty and has no root, then we consider the relation of having the same label of the root defined on $\mathcal{C} - r$, which partitions the set $\mathcal{C} - r$ into equivallence classes each of which has a root. Then $\mathcal{C}$ has as its immediate successors all of the equivallence classes (and only them). Obviously, the label of a set is its root.*

THEOREM 2. *Assume that Σ is an e-s-scenario for question Q relative to X. Then $\mathcal{T}(\phi^{-1}(\Sigma))$ is an e-t-scenario for question Q relative to X.*

Proof. Let $\Sigma = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ be an e-s-scenario and let $\phi^{-1}(\mathbf{s}_i) = \mathbf{b}_i$. We know (by the definition of ϕ) that $\phi^{-1}(\Sigma) = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a finite set of finite r-chains labelled with wffs. By Definition 1 (e-derivation), for each $\mathbf{b}_i$, its root is labelled with Q. Thus $\phi^{-1}(\Sigma)$ has a root and we may define the tree $\mathcal{T}(\phi^{-1}(\Sigma))$. We know already that clauses (1) and (2) of Definition 3 are satisfied. Definition 1 gives us also clause (3) of Definition 3 and clause (1) of Definition 2 warrants that (4) of Definition 3 is satisisfied. Clause (5) of Definition 3 is guaranteed by clause (2) of Definition 1, clause (6) of 3 — by (3b) of Definition 2 and by the construction of $\mathcal{T}(\phi^{-1}(\Sigma))$, clause (7) — by Corollary 3. And finally, clause (8) of Definition 3 is warranted by clause (1) of Definition 1, clause (3) of Definition 2 and the construction of $\mathcal{T}(\phi^{-1}(\Sigma))$. □

It is also nice to state that:

COROLLARY 8. *Let Σ be an e-s-scenario and let $\mathcal{B}$ be the set of all branches of $\mathcal{T}(\phi^{-1}(\Sigma))$. Then $\phi(\mathcal{B}) = \Sigma$.*

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