Permittance Erotetic Implication. Research Report

Research report number 2(3)/2014; published: May 29, 2014. This work is a part of the project titled *Erotetic logic in the modeling of ultimate and distributed question processing. Theoretical foundations and applications* supported by funds of the National Science Centre, Poland, (DEC-2012/04/A/HS1/00715).

Mariusz Urbański
Department of Logic and Cognitive Science
Institute of Psychology, Adam Mickiewicz University

2014
Poznań, Poland
www.kognitywistyka.amu.edu.pl/intquestpro

Permittance Erotetic Implication (Research Report)

Mariusz Urbański
Department of Logic and Cognitive Science
Institute of Psychology
Adam Mickiewicz University, Poznań, Poland
Mariusz.Urbanski@amu.edu.pl

1 Introduction

This research report presents some results on question raising as modelled in the logic of permittance (Wiśniewski [2013]). The first section of the report offers some theorems concerning basic semantic notions of this logic: permittance, transmission of permittance, logical entailment, and epistemic modality $\boxplus$ (of "knowledge in a state"). In the second section some paradoxes of irrelevant permittance evocation of questions are given. The third section presents examples of two versions of permittance erotetic implication, and some theorems on their relationships.

The report is not self-contained: for all the notions and symbols used here the reader should refer to Wiśniewski [2013].

2 $\boxplus$-fixing

Definition 1. *Let Y be a set of formulas of L. $\boxplus Y = \{\boxplus A : A \in Y\}$.*

Definition 2. *Let Q be a question. $\boxplus Q$ is a question such that:*

$$\mathbf{d} \boxplus Q = \boxplus \mathbf{d} Q$$

Lemma 1. *Let $\mathbf{M}$ be an L-model and σ be an $\mathbf{M}$-state. If $\sigma \looparrowright \boxplus X$, then $\sigma \looparrowright X$.*

Proof. Suppose that $\sigma \looparrowright \boxplus X$ and that $\sigma \not\looparrowright X$. Then for some $A \in X$: $\sigma \not\looparrowright A$. There are two cases to be considered:

1. A is a p-wff. In this case, for any $w \in \sigma$, $\mathbf{M}, w \not\models A$. On the other hand, $\sigma \looparrowright \boxplus A$ and thus for any $w \in \sigma$, $\mathbf{M}, w \models A$. A contradiction.

2. A is a n-wff. In this case, there exists $w \in \sigma$ such that $\mathbf{M}, w \not\models A$. On the other hand, $\sigma \looparrowright \boxplus A$ and thus for any $w \in \sigma$, $\mathbf{M}, w \models A$. A contradiction. □

The converse of lemma 1 does not hold. Consider an L-model $\mathbf{M} = \langle \{w_1, w_2\}, \mathcal{V} \rangle$ and its state $\sigma = \{w_1, w_2\}$ such that $\mathcal{V}(p, w_1) = \mathbf{1}$ and $\mathcal{V}(p, w_2) = \mathbf{0}$. In this case $\sigma \looparrowright \{p\}$ while $\sigma \not\looparrowright \boxplus\{p\}$.

Recall that (Wiśniewski [2013]):

Corollary 6. *Let $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ be a L-model and $\{w\}$ be a (singleton) $\mathbf{M}$-state. Then $\{w\} \looparrowright A$ iff $\mathbf{M}, w \models A$.*

The following holds:

Corollary 1. *Let $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ be a L-model and $\{w\}$ be a (singleton) $\mathbf{M}$-state. Then $\{w\} \looparrowright A$ iff $\{w\} \looparrowright \boxplus A$.*

Proof. ($\Rightarrow$) Suppose that $\{w\} \looparrowright A$. Thus for all elements of the considered state $\mathbf{M}, w \models A$. As a result, $\{w\} \looparrowright \boxplus A$.
($\Leftarrow$) By lemma 1. □

An analogue of corollary 1 holds for $\oplus$. For $\ominus$ holds the following: $\{w\} \looparrowright \neg A$ iff $\{w\} \looparrowright \ominus A$.

Lemma 2. *If $X \hookrightarrow_L \boxplus A$, then $X \hookrightarrow_L A$.*

Proof. Suppose that $X \hookrightarrow_L \boxplus A$. Thus for each L-model $\mathbf{M}$ and each $\mathbf{M}$-state σ: if $\sigma \looparrowright X$, then $\sigma \looparrowright \boxplus A$. By lemma 1, if $\sigma \looparrowright \boxplus A$, then $\sigma \looparrowright A$. As a result, $X \hookrightarrow_L A$. □

The converse of lemma 2 does not hold.

Lemma 3. *$X \models_L A$ iff $\boxplus X \models_L A$.*

Proof. ($\Rightarrow$) Suppose that $X \models_L A$ and that $\boxplus X \not\models_L A$. Thus there exists an L-model $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ and $w \in \mathcal{W}$ such that for any $B \in X$, $\mathbf{M}, w \models \boxplus B$ and $\mathbf{M}, w \not\models A$. Thus for any $B \in X$, $\mathbf{M}, w \models \neg(B \to \bot)$ and $\mathbf{M}, w \not\models B \to \bot$. As a result, $\mathbf{M}, w \models B$. As this holds for any $B \in X$, $X \not\models_L A$. A contradiction.
($\Leftarrow$) Suppose that $\boxplus X \models_L A$ and that $X \not\models_L A$. Thus there exists an L-model $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ and $w \in \mathcal{W}$ such that for any $B \in X$, $\mathbf{M}, w \models B$ and $\mathbf{M}, w \not\models A$. Consider an L-model $\mathbf{M}' = \langle \{w\}, \mathcal{V} \rangle$. Clearly, $\mathbf{M}', w \not\models A$. For any $B \in X$, $\mathbf{M}', w \not\models B \to \bot$, and $\mathbf{M}', w \models \neg(B \to \bot)$, and $\mathbf{M}', w \models \boxplus B$. As a result, $\boxplus X \not\models_L A$. A contradiction. □

Lemma 4. *$X \models_L A$ iff $\boxplus X \hookrightarrow_L A$.*

Proof. ($\Rightarrow$) Suppose that $X \models_L A$ and that $\boxplus X \not\hookrightarrow_L A$. Thus there exists an L-model $\mathbf{M}$ and an $\mathbf{M}$-state σ such that $\sigma \looparrowright \boxplus X$ and $\sigma \not\looparrowright A$. There are two cases to be considered:

1. A is a p-wff. In this case, for any $w \in \sigma$, $\mathbf{M}, w \not\models A$. On the other hand, for any $B \in X$, $\sigma \looparrowright \boxplus B$ and thus for any $w \in \sigma$, $\mathbf{M}, w \models B$. As a result $X \not\models_L A$. A contradiction.

2. A is a n-wff. In this case, there exists $w \in \sigma$ such that $\mathbf{M}, w \not\models A$. On the other hand, for any $B \in X$, $\sigma \looparrowright \boxplus B$ and thus for any $w' \in \sigma$, $\mathbf{M}, w' \models B$. As a result $X \not\models_L A$. A contradiction.

($\Leftarrow$) Suppose that $\boxplus X \hookrightarrow_L A$ and $X \not\models_L A$. Thus there exists an L-model $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ and $w \in \mathcal{W}$ such that for any $B \in X$, $\mathbf{M}, w \models B$ and $\mathbf{M}, w \not\models A$. Now, consider a singleton $\mathbf{M}$-state $\{w\}$. Clearly, for any $B \in X$, $\{w\} \looparrowright B$. By corollary 1, $\{w\} \looparrowright \boxplus B$ and thus $\{w\} \looparrowright \boxplus X$ while $\{w\} \not\looparrowright A$. Thus $\boxplus X \not\hookrightarrow_L A$. A contradiction. □

Recall that (Wiśniewski [2013]):

Corollary 11. *If* $X \hookrightarrow_L A$, *then* $X \models_L A$.

Corollary 2. *If* $X \hookrightarrow_L \boxplus A$, *then* $X \models_L A$.

Proof. By lemma 2 and corollary 11. □

Lemma 5. *If* $X \hookrightarrow_L \boxplus A$, *then* $\boxplus X \hookrightarrow_L A$.

Proof. Suppose that $X \hookrightarrow_L \boxplus A$ and that $\boxplus X \not\hookrightarrow_L A$. Thus there exists an L-model $\mathbf{M}$ and an $\mathbf{M}$-state σ such that $\sigma \looparrowright \boxplus X$ and $\sigma \not\looparrowright A$. By lemma 1, $\sigma \looparrowright X$. There are two cases to be considered:

1. A is a p-wff. In this case, for any $w \in \sigma$, $\mathbf{M}, w \not\models A$, and $\mathbf{M}, w \not\models \boxplus A$. Thus $X \not\hookrightarrow_L \boxplus A$. A contradiction.

2. A is a n-wff. In this case, there exists $w \in \sigma$ such that $\mathbf{M}, w \not\models A$. Thus $\mathbf{M}, w \not\models \boxplus A$ and $X \not\hookrightarrow_L \boxplus A$. A contradiction. □

Lemma 5 is a consequence of corollary 2 and lemma 4. Notice also that lemma 2 might be proven by lemmata 1 and 5.

Lemma 6. *If* $X \hookrightarrow_L A$, *then* $\boxplus X \hookrightarrow_L A$.

Proof. Suppose that $X \hookrightarrow_L A$ and that $\boxplus X \not\hookrightarrow_L A$. Thus there exists an L-model $\mathbf{M}$ and an $\mathbf{M}$-state σ such that $\sigma \looparrowright \boxplus X$ and $\sigma \not\looparrowright A$. By lemma 1, $\sigma \looparrowright X$. Therefore $X \not\hookrightarrow_L A$. A contradiction. □

Lemma 6 is a consequence of corollary 11 and lemma 4. Its converse does not hold: consider $X = \{p \to q, p\}$ and $A = q$.

Lemma 7. *If* $X \hookrightarrow_L A$, *then* $\boxplus X \hookrightarrow_L \boxplus A$.

Proof. Suppose that $X \hookrightarrow_L A$ and that $\boxplus X \not\hookrightarrow_L \boxplus A$. Thus there exists an L-model $\mathbf{M}$ and an $\mathbf{M}$-state σ such that $\sigma \looparrowright \boxplus X$ and $\sigma \not\looparrowright \boxplus A$. For each $B \in X$, $\sigma \looparrowright \boxplus B$. Thus for each $w \in \sigma$, $\mathbf{M}, w \models B$. On the other hand, there exists $w' \in \sigma$ such that $\mathbf{M}, w' \not\models A$. Now, consider $\mathbf{M}$-state $\sigma' = \{w'\}$. Clearly, $\sigma \looparrowright X$ and $\sigma \not\looparrowright A$, thus $X \not\hookrightarrow_L A$. A contradiction. □

3 Paradoxes of irrelevant permittance question evocation

Evocation of questions forms an explication of an intuitive notion of arising of a question from (or on the basis of) some set of declarative sentences. In the context of the logic of permittance it is defined by:

Definition 3. $\mathbf{E}_{\looparrowright}(X, Q)$ *iff*

1. $X \| \!\!\hookrightarrow_L \mathbf{d}Q$*, and*
2. $X \not\hookrightarrow_L A$*, for any* $A \in \mathbf{d}Q$*.*

For any X the following hold, provided that the second clause of the definition of $\looparrowright$-evocation is met:

$$\mathbf{E}_{\looparrowright}(X, ?\{\oplus p, \ominus p\}) \tag{1}$$

$$\mathbf{E}_{\looparrowright}(X, ?\{p, \neg p\}) \tag{2}$$

$$\mathbf{E}_{\looparrowright}(X, ?\{p, \ominus p\}) \tag{3}$$

$$\mathbf{E}_{\looparrowright}(X, ?\{\oplus p, \neg p\}) \tag{4}$$

Examples 1 – 4 seem somewhat paradoxical. They show that some questions arise from any set X of declaratives, provided that no answer to those questions is entailed by X. It is possible, then, that X evokes a question Q while answers to Q are irrelevant from the point of view of X (e.g. formulas in X and in $\mathbf{d}Q$ do not share any variable). Questions in examples 1 – 4 might be interpreted as simple yes-no questions, but the same holds for each question such that a disjunction of its answers is a valid formula (and also for evocation defined on the basis of Classical Logic).

4 Permittance erotetic implication

Erotetic implication is an explication of an intuitive notion of arising of a question from (or on the basis of) a question and some set of declarative sentences. In Wiśniewski [2013] there are defined two versions of this notion (definitions 4 and 5).

Definition 4. $\mathbf{Im}_{\looparrowright}(Q, X, Q_1)$ *iff*

1. *for each* $A \in \mathbf{d}Q : X \cup \{A\} \| \!\!\hookrightarrow_L \mathbf{d}Q_1$*, and*
2. *for each* $B \in \mathbf{d}Q_1$ *there exists a non-empty proper subset* Y *of* Q *such that* $X \cup \{B\} \| \!\!\hookrightarrow_L Y$*.*

Some examples:

$$\mathbf{Im}_{\looparrowright}(?\boxplus p, p, ?\odot p) \tag{5}$$

$$\mathbf{Im}_{\looparrowright}(?\odot p, p, ?\boxplus p) \tag{6}$$

$$\mathbf{Im}_{\looparrowright}(?\boxplus p, \neg p, ?\odot p) \tag{7}$$

$$\mathbf{Im}_{\looparrowright}(?\odot p, \neg p, ?\boxplus p) \tag{8}$$

$$\mathbf{Im}_{\looparrowright}(?\boxplus p, p, ?\ominus p) \tag{9}$$

$$\mathbf{Im}_{\looparrowright}(?\ominus p, p, ?\boxplus p) \tag{10}$$

$$\mathbf{Im}_{\looparrowright}(?\boxplus p, \neg p, ?\ominus p) \tag{11}$$

$$\mathbf{Im}_{\looparrowright}(?\ominus p, \neg p, ?\boxplus p) \tag{12}$$

$$\mathbf{Im}_{\looparrowright}(?\boxplus p, \neg p, ?\oplus p) \tag{13}$$

$$*\mathbf{Im}_{\looparrowright}(?\oplus p, \neg p, ?\boxplus p) \tag{14}$$

$$*\mathbf{Im}_{\looparrowright}(?\boxplus p, p, ?\oplus p) \tag{15}$$

$$\mathbf{Im}_{\looparrowright}(?\oplus p, p, ?\boxplus p) \tag{16}$$

17 – 19 are examples of pure $\looparrowright$-erotetic implication (Wiśniewski [1995], p. 183). 17 and 18 are also *analytic* (Wiśniewski [2001], p. 27), while 19 is not.

$$\mathbf{Im}_{\looparrowright}(?\{A, B\}, \emptyset, ?\{A, B\}) \tag{17}$$

where A, B are formulas of L (no matter if p-formulas or n-formulas).

$$\mathbf{Im}_{\looparrowright}(?\{p, \ominus p\}, \emptyset, ?p) \tag{18}$$

$$\mathbf{Im}_{\looparrowright}(?p, \emptyset, ?\{p, \ominus p\}) \tag{19}$$

In the example 20 what holds is $\looparrowright$-erotetic implication as well as its *strong* version (Wiśniewski [1995], p. 187). In the case of example 21 the strong version does not hold.

$$\mathbf{Im}_{\looparrowright}(?(p \vee q), \neg q, ?p) \tag{20}$$

$$\mathbf{Im}_{\hookrightarrow}(?(p \vee q), q, ?p) \tag{21}$$

For conjunction the analogue to 20 holds (example 22; and this implication is also not strong), while 23, the analogue to 21, does not:

$$\mathbf{Im}_{\hookrightarrow}(?(p \wedge q), \neg q, ?p) \tag{22}$$

$$*\mathbf{Im}_{\hookrightarrow}(?(p \wedge q), q, ?p) \tag{23}$$

These do not hold:

$$*\mathbf{Im}_{\hookrightarrow}(?\{p, q, r\}, s \vee t, s \to p, t \to q \vee r, ?\{s, t\}) \tag{24}$$

For, consider an L-model $\mathbf{M} = \langle \{w_1, w_2\}, \mathcal{V} \rangle$ and its state $\sigma = \{w_1, w_2\}$ such that: $\mathcal{V}(s, w_2) = \mathbf{1}$ and $\mathcal{V}(p, w_1) = \mathcal{V}(q, w_1) = \mathcal{V}(r, w_1) = \mathcal{V}(p, w_2) = \mathcal{V}(q, w_2) = \mathcal{V}(r, w_2) = \mathcal{V}(s, w_2) = \mathcal{V}(t, w_2) = \mathbf{0}$.

$$*\mathbf{Im}_{\hookrightarrow}(?\{p, q, r\}, s \vee t, \boxplus(s \to p), \boxplus(t \to q \vee r), ?\{s, t\}) \tag{25}$$

$$*\mathbf{Im}_{\hookrightarrow}(?\{\boxplus p, \boxplus q, \boxplus r\}, s \vee t, s \to p, t \to q \vee r, ?\{s, t\}) \tag{26}$$

$$*\mathbf{Im}_{\hookrightarrow}(?\{\boxplus p, \boxplus q, \boxplus r\}, s \vee t, s \to p, t \to q \vee r, ?\{\boxplus s, \boxplus t\}) \tag{27}$$

$$*\mathbf{Im}_{\hookrightarrow}(?\{\boxplus p, \boxplus q, \boxplus r\}, \boxplus(s \vee t), \boxplus(s \to p), \boxplus(t \to q \vee r), ?\{\boxplus s, \boxplus t\}) \tag{28}$$

$$*\mathbf{Im}_{\hookrightarrow}(?\{p, q, r\}, \boxplus(s \vee t), \boxplus(s \to p), \boxplus(t \to q \vee r), ?\{\boxplus s, \boxplus t\}) \tag{29}$$

$$*\mathbf{Im}_{\hookrightarrow}(?\{p, q, r\}, \odot(s \vee t), s \to p, t \to q \vee r, ?\{\odot s, \odot t\}) \tag{30}$$

$$*\mathbf{Im}_{\hookrightarrow}(?\{p, q, r\}, \oplus(s \vee t), s \to p, t \to q \vee r, ?\{\oplus s, \oplus t\}) \tag{31}$$

$$*\mathbf{Im}_{\hookrightarrow}(?\{p, q, r\}, \oplus s \vee \oplus t, s \to p, t \to q \vee r, ?\{\oplus s, \oplus t\}) \tag{32}$$

But these hold:

$$\mathbf{Im}_{\hookrightarrow}(?\{p, q, r\}, \boxplus(s \vee t), \boxplus(s \to p), \boxplus(t \to q \vee r), ?\{s, t\}) \tag{33}$$

$$\mathbf{Im}_{\hookrightarrow}(?\{p, q, r\}, \boxplus(s \vee t), s \to p, t \to q \vee r, ?\{s, t\}) \tag{34}$$

The premise $\boxplus(s \vee t)$ makes the question $?\{s, t\}$ relatively safe (with respect to a state). Notice that implications of examples 5 – 21 are *regular* (Wiśniewski [2001], p. 26), while implications of examples 33 and 34 are not.

Theorem 1. *If* $\mathbf{Im}_{\looparrowright}(Q, X, Q_1)$*, then* $\mathbf{Im}_{\looparrowright}(Q, \boxplus X, Q_1)$.

Proof. Assume that (*) $\mathbf{Im}_{\looparrowright}(Q, X, Q_1)$.

Suppose that there exists $A \in \mathbf{d}Q$ such that $\boxplus X \cup \{A\} \| \not\hookrightarrow_L \mathbf{d}Q_1$. Thus there exists an L-model $\mathbf{M}$ and its state σ such that $\sigma \looparrowright \boxplus X \cup \{A\}$ and for any $w \in \sigma$: $\mathbf{M}, w \not\models B$, for any $B \in \mathbf{d}Q_1$. Therefore $\sigma \looparrowright \boxplus X$ and, by lemma 1, $\sigma \looparrowright X$. Thus $\sigma \looparrowright X \cup \{A\}$ and $X \cup \{A\} \| \not\hookrightarrow_L \mathbf{d}Q_1$. As a result, (*) does not hold. A contradiction.

Suppose that there exists $B \in \mathbf{d}Q_1$ such that for any non-empty proper subset Y of $\mathbf{d}Q$: $\boxplus X \cup \{B\} \| \not\hookrightarrow_L \mathbf{d}Q$. Thus there exists an L-model $\mathbf{M}$ and its state σ such that $\sigma \looparrowright \boxplus X \cup \{B\}$ and for any $w \in \sigma$: $\mathbf{M}, w \not\models C$, for any $C \in Y$. Therefore $\sigma \looparrowright \boxplus X$ and, by lemma 1, $\sigma \looparrowright X$. Thus $\sigma \looparrowright X \cup \{B\}$ and $X \cup \{B\} \| \not\hookrightarrow_L Y$. As a result, (*) does not hold. A contradiction. □

The converse of theorem 1 does not hold. For, consider examples 33 and 24.

Definition 5. $\mathbf{Im}'_{\looparrowright}(Q, X, Q_1)$ *iff*

1. *for each* $A \in \mathbf{d}Q : X \cup \{A\} \| \hookrightarrow_L \mathbf{d}Q_1$*, and*
2. *for each* $B \in \mathbf{d}Q_1$ *there exists a non-empty proper subset* Y *of* Q *such that* $X \cup \{B\} \| \hookrightarrow_L \boxplus Y$.

For example:

$$*\mathbf{Im}'_{\looparrowright}(?\{p, q, r\}, \boxplus(s \vee t), \boxplus(s \to p), \boxplus(t \to q \vee r), ?\{\boxplus s, \boxplus t\}) \tag{35}$$

$$*\mathbf{Im}'_{\looparrowright}(?\{\boxplus p, \boxplus q, \boxplus r\}, \boxplus(s \vee t), \boxplus(s \to p), \boxplus(t \to q \vee r), ?\{\boxplus s, \boxplus t\}) \tag{36}$$

$$\mathbf{Im}'_{\looparrowright}(?\{\boxplus A, \boxplus B\}, \emptyset, ?\{\boxplus A, \boxplus B\}) \tag{37}$$

In example 37 A, B are formulas of L (no matter if p-formulas or n-formulas).

$$\mathbf{Im}'_{\looparrowright}(?\{A, B\}, \emptyset, ?\{\boxplus A, \boxplus B\}) \tag{38}$$

$$\mathbf{Im}'_{\looparrowright}(?\{A, B\}, \emptyset, ?\{A, B\}) \tag{39}$$

In examples 38 and 39 A, B are n-formulas of L.

$$*\mathbf{Im}'_{\looparrowright}(?\{p, \ominus p\}, \emptyset, ?\{\boxplus p, \boxplus \neg p\}) \tag{40}$$

$$\mathbf{Im}'_{\looparrowright}(?\{p, \ominus p\}, \emptyset, ?\{\boxplus p, \neg \boxplus p\}) \tag{41}$$

$$\mathbf{Im}'_{\looparrowright}(?\{p, \ominus p\}, \emptyset, ?\{\boxplus p, \boxplus \ominus p\}) \tag{42}$$

$$*\mathbf{Im}'_{\looparrowright}(?p, \emptyset, ?\{\boxplus p, \boxplus \ominus p\}) \tag{43}$$

$$\mathbf{Im}'_{\looparrowright}(?(p \wedge q), \neg q, ?p) \tag{44}$$

$$*\mathbf{Im}'_{\looparrowright}(?(p \vee q), \neg q, ?p) \tag{45}$$

$$\mathbf{Im}'_{\looparrowright}(?(p \wedge q), \neg q, ?\{\boxplus p, \neg \boxplus p\}) \tag{46}$$

$$*\mathbf{Im}'_{\looparrowright}(?(p \vee q), \neg q, ?\{\boxplus p, \neg \boxplus p\}) \tag{47}$$

Notice that erotetic implications of examples 44 and 46 are not strong.

$$*\mathbf{Im}'_{\looparrowright}(?(p \# q), q, ?p) \tag{48}$$

$$*\mathbf{Im}'_{\looparrowright}(?(p \# q), q, ?\{\boxplus p, \neg \boxplus p\}) \tag{49}$$

$$*\mathbf{Im}'_{\looparrowright}(?(p \# q), \neg q, ?\{\boxplus p, \boxplus \neg p\}) \tag{50}$$

$$*\mathbf{Im}'_{\looparrowright}(?(p \# q), q, ?\{\boxplus p, \boxplus \neg p\}) \tag{51}$$

The symbol # stands for either $\vee$ or $\wedge$.

Theorem 2. *If* $\mathbf{Im}'_{\looparrowright}(Q, X, Q_1)$*, then* $\mathbf{Im}_{\looparrowright}(Q, X, Q_1)$.

Proof. Assume that (*) $\mathbf{Im}'_{\looparrowright}(Q, X, Q_1)$. As first clauses of definitions 4 and 5 are identical, we shall consider only the second clauses.

Suppose that there exists $B \in \mathbf{d}Q_1$ such that for any non-empty proper subset Y of $\mathbf{d}Q$: $X \cup \{B\} \| \not\rightarrow_L Y$. Thus there exists an L-model $\mathbf{M}$ and its state σ such that $\sigma \looparrowright \boxplus X \cup \{B\}$ and for any $w \in \sigma$: $\mathbf{M}, w \not\models C$, for any $C \in Y$. If so, then $\mathbf{M}, w \not\models \boxplus C$. As this is the case for each $w \in \sigma$ and for each $C \in Y$, $X \cup \{B\} \| \not\rightarrow_L \boxplus Y$. Thus (*) does not hold. A contradiction. □

The converse of theorem 2 does not hold. For, consider example 34 and an L-model $\mathbf{M} = \langle \{w_1, w_2\}, \mathcal{V} \rangle$ and its state $\sigma = \{w_1, w_2\}$ such that: $\mathcal{V}(p, w_1) = \mathcal{V}(s, w_1) = \mathcal{V}(q, w_2) = \mathcal{V}(t, w_2) = \mathbf{1}$ while $\mathcal{V}(q, w_1) = \mathcal{V}(r, w_1) = \mathcal{V}(t, w_1) = \mathcal{V}(p, w_2) = \mathcal{V}(s, w_2) = \mathbf{0}$.

Theorem 3. *If* $\mathbf{Im}'_{\looparrowright}(Q, X, Q_1)$*, then* $\mathbf{Im}_{\looparrowright}(\boxplus Q, X, Q_1)$.

Proof. Assume that (*) $\mathbf{Im}'_{\looparrowright}(Q, X, Q_1)$.

Suppose that there exists $A \in \mathbf{d}Q$ such that $X \cup \{\boxplus A\} \| \not\rightarrow_L \mathbf{d}Q_1$. Thus there exists an L-model $\mathbf{M}$ and its state σ such that $\sigma \looparrowright X \cup \{\boxplus A\}$ and for any $w \in \sigma$: $\mathbf{M}, w \not\models B$, for any $B \in \mathbf{d}Q_1$. Therefore $\sigma \looparrowright \boxplus A$ and, by lemma 1,

$\sigma \looparrowright A$. Thus $\sigma \looparrowright X \cup \{A\}$ and $X \cup \{A\} \| \not\to_L \mathbf{d}Q_1$. As a result, (*) does not hold. A contradiction.

Suppose that there exists $B \in \mathbf{d}Q_1$ such that for any non-empty proper subset Y of $\mathbf{d} \boxplus Q$: $X \cup \{B\} \| \not\to_L Y$. Notice that $Y = \boxplus Z$, where Z is a non-empty proper subset of $\mathbf{d}Q$. Thus there exists $B \in \mathbf{d}Q_1$ such that for any non-empty proper subset Z of $\mathbf{d}Q$: $X \cup \{B\} \| \not\to_L \boxplus Z$. As a result, (*) does not hold. A contradiction. □

The converse of theorem 3 does not hold.

Lemma 8. $\sigma \looparrowright \boxplus A$ *iff* $\sigma \looparrowright \boxplus \boxplus A$.

Lemma 9. $\mathbf{M}, w \models \boxplus A$ *iff* $\mathbf{M}, w \models \boxplus \boxplus A$.

Theorem 4. $\mathbf{Im}_{\looparrowright}(\boxplus Q, X, Q_1)$ *iff* $\mathbf{Im}'_{\looparrowright}(\boxplus Q, X, Q_1)$.

Proof. As first clauses of definitions 4 and 5 are identical, we shall consider only the second clauses.

($\Rightarrow$) Assume that (*) $\mathbf{Im}_{\looparrowright}(\boxplus Q, X, Q_1)$.

Suppose that there exists $B \in \mathbf{d}Q_1$ such that for any non-empty proper subset Y of $\mathbf{d} \boxplus Q$: $X \cup \{B\} \| \not\to_L \boxplus Y$. Thus there exists an L-model $\mathbf{M}$ and an $\mathbf{M}$-state σ such that: $\sigma \looparrowright X \cup B$ and for any $w \in \sigma$: $\mathbf{M}, w \not\models C$ for any $C \in \boxplus Y$. Notice, that each $C \in \boxplus Y$ is of the form $\boxplus \boxplus D$, where $\boxplus D \in Y$. By lemma 9, for any $w \in \sigma$: $\mathbf{M}, w \not\models \boxplus D$. Thus for any non-empty proper subset Y of $\mathbf{d} \boxplus Q$: $X \cup \{B\} \| \not\to_L Y$. As a result, (*) does not hold. A contradiction.

($\Leftarrow$) Assume that (*) $\mathbf{Im}'_{\looparrowright}(\boxplus Q, X, Q_1)$.

Suppose that there exists $B \in \mathbf{d}Q_1$ such that for any non-empty proper subset Y of $\mathbf{d} \boxplus Q$: $X \cup \{B\} \| \not\to_L Y$. Thus there exists an L-model $\mathbf{M}$ and an $\mathbf{M}$-state σ such that: $\sigma \looparrowright X \cup B$ and for any $w \in \sigma$: $\mathbf{M}, w \not\models C$ for any $C \in Y$. Notice, that each $C \in Y$ is of the form $\boxplus D$ and that $\boxplus \boxplus D \in \boxplus Y$. By lemma 9, for any $w \in \sigma$: $\mathbf{M}, w \not\models \boxplus \boxplus D$. Thus for any non-empty proper subset Y of $\mathbf{d} \boxplus Q$: $X \cup \{B\} \| \not\to_L \boxplus Y$. As a result, (*) does not hold. A contradiction. □

References

Wiśniewski, A. [1995]. *The Posing of Questions: Logical Foundations of Erotetic Inferences.* Kluwer Academic Publishers, Dordrecht/Boston/London.

Wiśniewski, A. [2001]. Questions and Inferences. *Logique et Analyse*, 173-174-175:5–43.

Wiśniewski, A. [2013]. Permittance, Inconsistencies, and Question Raising. Manuscript.