Identifying minimal truth conditions and minimal synthetic tableaux for CPC formulas. Research Report

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Identifying minimal truth conditions and minimal synthetic tableaux for CPC formulas

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Abstract

In this paper I shall give a method of identifying minimal truth conditions and minimal synthetic tableaux for a given formula A of Classical Propositional Calculus, via reduction of clausal Disjunctive Normal Form of A.

I shall consider Classical Propositional Calculus with $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (implication) as primitive connectives and with usual syntax and semantics. A *literal* is a propositional variable or the negation of a propositional variable. Literals of the form p_i, $\neg p_i$ are *complementary* literals. A literal complementary to a literal φ will be represented by φ'.

Let $(B_{1_1} \wedge \ldots \wedge B_{1_n}) \vee \ldots \vee (B_{i_1} \wedge \ldots \wedge B_{i_m})$, where $i, n, m \geq 1$, be a DNF of a formula A. The set $\{\{B_{1_1}, \ldots, B_{n_1}\}, \ldots, \{B_{i_1}, \ldots, B_{i_m}\}\}$ is a *clausal DNF* (cDNF) of A, where the sets $\{B_{1_1}, \ldots, B_{n_1}\}, \ldots, \{B_{i_1}, \ldots, B_{i_m}\}$ are (conjunctive) *clauses* of cDNF of A. Thus cDNF of A is a set of sets made up of literals of conjunctions of DNF of A.

A clause $\{\varphi_1, \ldots, \varphi_n\}$ is *true* under a valuation $\mathbf{v}$ iff $\mathbf{v}(\varphi_i) = \mathbf{1}$, for all $1 \leq i \leq n$. Otherwise $\{\varphi_1, \ldots, \varphi_n\}$ is *false* under $\mathbf{v}$. A clause $\{\varphi_1, \ldots, \varphi_n\}$ is *satisfiable* iff there exists a valuation $\mathbf{v}$ such that the clause is true under $\mathbf{v}$; otherwise the clause is *unsatisfiable.*

Corollary 1. *A clause is unsatisfiable iff it contains at least one pair of complementary literals.*

A cDNF of A is said to contain *complete contradictory clauses* iff it contains clauses of the form $\{\varphi_1, \ldots, \varphi_n\}, \{\varphi'_1\}, \ldots \{\varphi'_n\}$. The clause $\{\varphi_1, \ldots, \varphi_n\}$ will be called a *conjunctive component* of cDNF of A and the clauses $\{\varphi'_1\}$, …, $\{\varphi'_n\}$ will be called a *disjunctive component* of cDNF of A.

Theorem 1. *A formula A is unsatisfiable iff its cDNF contains unsatisfiable clauses only.*

Corollary 2. *A formula A is valid iff cDNF of $\neg A$ contains unsatisfiable clauses only.*

Reduction of cDNF of A consists of the following steps:

(RU) remove from cDNF of A all the unsatisfiable clauses (if any);

(RS) if there exist in cDNF of A clauses α_1, α_2 such that $\alpha_1 \subset \alpha_2$, then remove the clause α_2;

(RC) if cDNF of A contains complete contradictory clauses, then remove the remaining clauses (if any).

Result of this reduction will be called *reduced* cDNF of A. If none of RU, RS, RC is applicable, then reduced cDNF form of A equals cDNF of A.

Theorem 2. *Let α_1, α_2 be sets of clauses such that α_2 results from α_1 by application of one of the RI, RS, RC. If α_1 is satisfiable, then so is α_2.*

Theorem 3. *Let α be a clause of a cDNF of a formula A. If α is satisfiable, then there exists a synthetic inference* s *of A such that all the elements of* s *are derived on the basis of literals of α.*

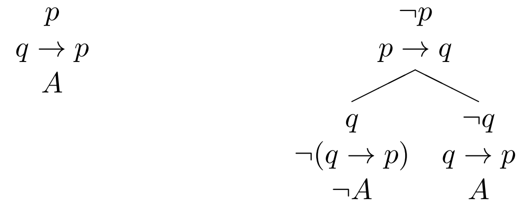
Corollary 3. *Let α be a clause of a reduced cDNF of a formula A. If α is satisfiable, then there exists a synthetic inference* s *of A such that all the elements of* s *are derived on the basis of literals of α.*

Example 1. *Synthetic tableau for $A = (p \to q) \to (q \to p)$*

for A:
DNF of A: $(p \wedge \neg q) \vee \neg q \vee p$
cDNF of A: $\{\{p, \neg q\}, \{\neg q\}, \{p\}\}$
reduced cDNF of A (by RS): $\{\{\neg q\}, \{p\}\}$

for $\neg A$:
DNF of $\neg A$: $(\neg p \wedge q \wedge \neg p) \vee (q \wedge q \wedge \neg p)$
cDNF of $\neg A$: $\{\{\neg p, q\}\}$
reduced cDNF of $\neg A$: $\{\{\neg p, q\}\}$

Sum of reduced cDNFs for A: $\{\{\neg q\}, \{p\}, \{\neg p, q\}\}$
Synthetic Ω tableau for A:



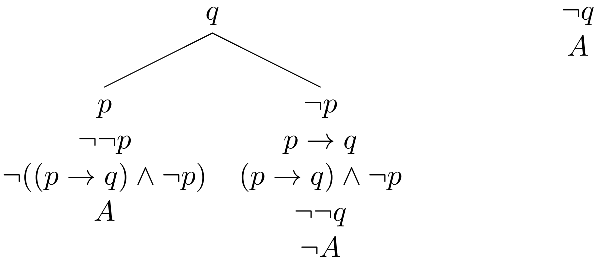
Sets of entangled literals on individual branches (left to right): $\{p\}$, $\{\neg p, q\}$, $\{\neg q\}$.
Thus sum of reduced cDNFs for A is made up of sets of entangled literals on branches of Ω.

Example 2. *Synthetic tableau for* $A = (p \to q) \wedge \neg p \to \neg q$

for A:
DNF of A: $(p \wedge \neg q) \vee p \vee \neg q$
cDNF of A: $\{\{p, \neg q\}, \{p\}, \{\neg q\}\}$
reduced cDNF of A (by RS): $\{\{p\}, \{\neg q\}\}$

for $\neg A$*:*
DNF of $\neg A$*:* $(\neg p \wedge \neg p \wedge q) \vee (q \wedge \neg p \wedge q)$
cDNF of $\neg A$*:* $\{\{\neg p, q\}\}$
reduced cDNF of $\neg A$*:* $\{\{\neg p, q\}\}$

Sum of reduced cDNFs for A: $\{\{p\}, \{\neg q\}, \{\neg p, q\}\}$
Synthetic Ω *tableau for A:*



Sets of entangled literals on individual branches (left to right): $\{p\}$, $\{\neg p, q\}$, $\{\neg q\}$.
Thus sum of reduced cDNFs for A is made up of sets of entangled literals on branches of Ω.

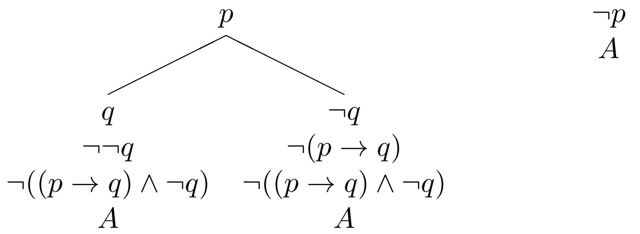
Example 3. *Synthetic tableau for* $A = (p \to q) \wedge \neg q \to \neg p$

for A:
DNF of A: $(p \wedge \neg q) \vee q \vee \neg p$
cDNF of A: $\{\{p, \neg q\}, \{q\}, \{\neg p\}\}$
reduced cDNF of A: $\{\{p, \neg q\}, \{q\}, \{\neg p\}\}$

for $\neg A$*:*
DNF of $\neg A$*:* $(\neg p \wedge \neg q \wedge p) \vee (q \wedge \neg q \wedge p)$
cDNF of $\neg A$*:* $\{\{\neg p, \neg q, p\}, \{q, \neg q, p\}\}$
reduced cDNF of $\neg A$ *(by RU):* $\{\emptyset\}$

Sum of reduced cDNFs for A: $\{\{p, \neg q\}, \{q\}, \{\neg p\}\}$

Synthetic Ω tableau for A:



Sets of entangled literals on individual branches (left to right): $\{q\}$, $\{p, \neg q\}$, $\{\neg p\}$.
Thus sum of reduced cDNFs for A is made up of sets of entangled literals on branches of Ω.

Example 4. *Synthetic tableau for $A = p \to (q \to p)$*

for A:
DNF of A: $\neg p \vee q \vee p$
cDNF of A: $\{\{\neg p\}, \{q\}, \{p\}\}$
reduced cDNF of A (by RC): $\{\{\neg p\}, \{p\}\}$

for $\neg A$:
DNF of $\neg A$: $p \wedge \neg q \wedge \neg p$
cDNF of $\neg A$: $\{\{p, \neg q, \neg p\}\}$
reduced cDNF of $\neg A$ (by RU): $\{\emptyset\}$

Sum of reduced cDNFs for A: $\{\{\neg p\}, \{p\}\}$
Synthetic Ω tableau for A:

p	$\neg p$
$q \to p$	A
A	

Sets of entangled literals on individual branches (left to right): $\{p\}$, $\{\neg p\}$.
Thus sum of reduced cDNFs for A is made up of sets of entangled literals on branches of Ω.

Example 5. *Synthetic tableau for $A = p \wedge r \to (q \to p)$*

for A:
DNF of A: $\neg p \vee \neg r \vee \neg q \vee p$
cDNF of A: $\{\{\neg p\}, \{\neg r\}, \{\neg q\}, \{p\}\}$
reduced cDNF of A (by RC): $\{\{\neg p\}, \{p\}\}$

for $\neg A$:
DNF of $\neg A$: $p \wedge r \wedge \neg q \wedge \neg p$
cDNF of $\neg A$: $\{\{p, r, \neg q, \neg p\}\}$

reduced cDNF of $\neg A$ *(by RU):* $\{\emptyset\}$

Sum of reduced cDNFs for A: $\{\{\neg p\}, \{p\}\}$
Synthetic Ω *tableau for A:*

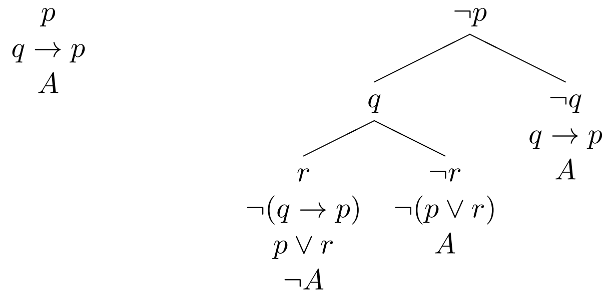
Sets of entangled literals on individual branches (left to right): $\{p\}$, $\{\neg p\}$.
Thus sum of reduced cDNFs for A is made up of sets of entangled literals on branches of Ω.

Example 6. *Synthetic tableau for* $A = p \vee r \to (q \to p)$

for A:
DNF of A: $(\neg p \wedge \neg r) \vee \neg q \vee p$
cDNF of A: $\{\{\neg p, \neg r\}, \{\neg q\}, \{p\}\}$
reduced cDNF of A: $\{\{\neg p, \neg r\}, \{\neg q\}, \{p\}\}$

for $\neg A$*:*
DNF of $\neg A$*:* $(p \wedge q \wedge \neg p) \vee (r \wedge q \wedge \neg p)$
cDNF of $\neg A$*:* $\{\{p, q, \neg p\}, \{r, q, \neg p\}\}$
reduced cDNF of $\neg A$ *(by RU):* $\{\{r, q, \neg p\}\}$

Sum of reduced cDNFs for A: $\{\{\neg p, \neg r\}, \{\neg q\}, \{p\}, \{r, q, \neg p\}\}$
Synthetic Ω *tableau for A:*



Sets of entangled literals on individual branches (left to right): $\{p\}$, $\{\neg p, q, r\}$, $\{\neg p, \neg r\}$, $\{\neg q\}$.
Thus sum of reduced cDNFs for A is made up of sets of entangled literals on branches of Ω.

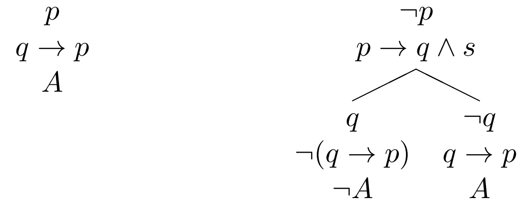
Example 7. *Synthetic tableau for* $A = (p \to q \wedge s) \to (q \to p)$

for A:
DNF of A: $(p \wedge \neg q) \vee (p \wedge \neg s) \vee \neg q \vee p$
cDNF of A: $\{\{p, \neg q\}, \{p, \neg s\}, \{\neg q\}, \{p\}\}$

reduced cDNF of A *(by RS):* $\{\{\neg q\}, \{p\}\}$

for $\neg A$*:*
DNF of $\neg A$*:* $(\neg p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge s \wedge q) \vee (\neg p \wedge q \wedge s)$
cDNF of $\neg A$*:* $\{\{\neg p, q\}, \{\neg p, q, s\}\}$
reduced cDNF of $\neg A$ *(by RS):* $\{\{\neg p, q\}\}$

Sum of reduced cDNFs for A*:* $\{\{\neg q\}, \{p\}, \{\neg p, q\}\}$
Synthetic Ω *tableau for* A*:*



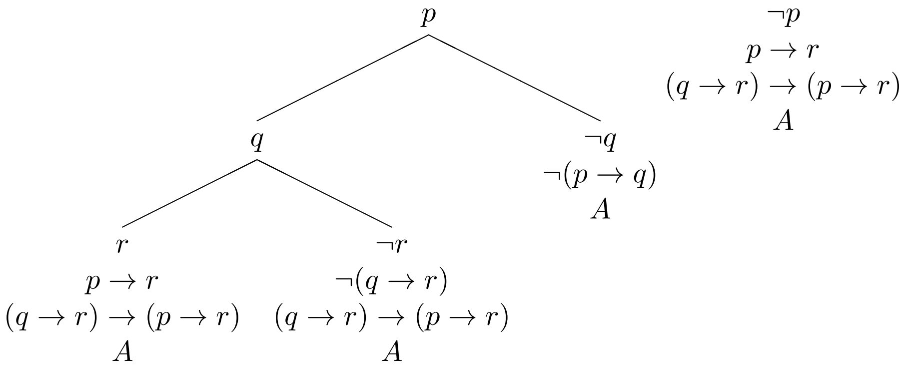
Sets of entangled literals on individual branches (left to right): $\{p\}$, $\{\neg p, q\}$, $\{\neg q\}$.
Thus sum of reduced cDNFs for A *is made up of sets of entangled literals on branches of* Ω.

Example 8. *Synthetic tableau for* $A = (p \to q) \to ((q \to r) \to (p \to r))$

for A*:*
DNF of A*:* $(p \wedge \neg q) \vee (q \wedge \neg r) \vee \neg p \vee r$
cDNF of A*:* $\{\{p, \neg q\}, \{q, \neg r\}, \{\neg p\}, \{r\}\}$
reduced cDNF of A*:* $\{\{p, \neg q\}, \{q, \neg r\}, \{\neg p\}, \{r\}\}$

for $\neg A$*:*
DNF of $\neg A$*:* $(\neg p \wedge \neg q \wedge p \wedge r) \vee (q \wedge \neg q \wedge p \wedge \neg r) \vee (\neg p \wedge r \wedge p \wedge \neg r) \vee (q \wedge r \wedge p \wedge \neg p)$
cDNF of $\neg A$*:* $\{\{\neg p, \neg q, p, r\}, \{q, \neg q, p, \neg r\}, \{\neg p, r, p, \neg r\}, \{q, r, p, \neg p\}\}$
reduced cDNF of $\neg A$ *(by RU):* $\{\emptyset\}$

Sum of reduced cDNFs for A*:* $\{\{p, \neg q\}, \{q, \neg r\}, \{\neg p\}, \{r\}\}$
Synthetic Ω *tableau for* A*:*



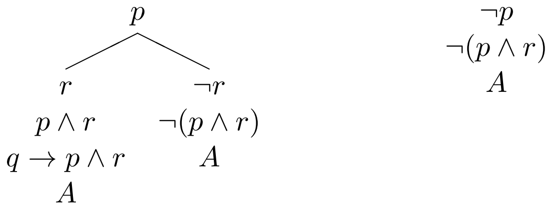
Sets of entangled literals on individual branches (left to right): $\{r\}$, $\{q, \neg r\}$, $\{p, \neg q\}$, $\{\neg p\}$.
Thus sum of reduced cDNFs for A is made up of sets of entangled literals on branches of Ω.

Example 9. *Synthetic tableau for* $A = p \wedge r \rightarrow (q \rightarrow p \wedge r)$

for A:
DNF of A: $\neg p \vee \neg r \vee \neg q \vee (p \wedge r)$
cDNF of A: $\{\{\neg p\}, \{\neg r\}, \{\neg q\}, \{p, r\}\}$
reduced cDNF of A (by RC): $\{\{\neg p\}, \{\neg r\}, \{p, r\}\}$

for $\neg A$*:*
DNF of $\neg A$*:* $(p \wedge r \wedge q \ \ \wedge \neg p) \vee (p \wedge r \wedge q \wedge \neg r)$
cDNF of $\neg A$*:* $\{\{p, r, q, \neg p\}, \{p, r, q, \neg r\}\}$
reduced cDNF of $\neg A$ *(by RU):* $\{\emptyset\}$

Sum of reduced cDNFs for A: $\{\{\neg p\}, \{\neg r\}, \{p, r\}\}$
Synthetic Ω *tableau for A:*



Sets of entangled literals on individual branches (left to right): $\{p, r\}$, $\{\neg r\}$, $\{\neg p\}$.
Thus sum of reduced cDNFs for A is made up of sets of entangled literals on branches of Ω.